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Duality results for co-compact Gabor systems

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Abstract—In this paper we give an account of recent developments in the duality theory of Gabor frames. We prove the Wexler-Raz biorthogonality relations and the duality principle for co-compact Gabor systems on second countable, locally compact abelian groups G . Our presentation does not rely on the existence of uniform lattices in G .

I. INTRODUCTION

In Gabor analysis the theory on duality results such as the Wexler-Raz biorthogonal relations and the duality principle plays an important role. The duality results go back to a series of papers in the 1990s [1]–[9] on discrete Gabor systems in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^d)$ with modulations and translations along lattices. Recently, these duality results were extended by the two authors [10] to the setting of Gabor systems in $L^2(G)$ with modulations and translations along closed, co-compact subgroups, where G is a second countable locally compact abelian (LCA) group. The aim of this note is to give an overview of these extended duality results. We mention that co-compact Gabor systems exist for any LCA group. On the other hand, there exist LCA groups that have no lattices, hence in these cases, the usual duality theory does not apply.

We will give a detailed account of co-compact Gabor systems in Section III and present the extended duality results in Section IV. In Section II we recall some key concepts from frame theory. Let us conclude this introduction with a simple description of co-compact Gabor systems in $L^2(\mathbb{R}^d)$.

Example 1. In $L^2(\mathbb{R}^d)$ co-compact Gabor systems are of the form

$$\{e^{2\pi i \gamma \cdot x} g(x - \lambda)\}_{\lambda \in A(\mathbb{R}^s \times \mathbb{Z}^{d-s}), \gamma \in B(\mathbb{R}^r \times \mathbb{Z}^{d-r})}$$

for some choice of $0 \leq r, s \leq d$, $g \in L^2(\mathbb{R}^d)$ and $A, B \in GL_d(\mathbb{R})$. Depending on the parameters r and s , these Gabor systems range from discrete over semi-continuous to continuous families. If we take $r = s = 0$, we recover the usual discrete *lattice* Gabor systems, while the choice $r = s = d$ yields the Gabor system associated with the *short-time Fourier transform*.

II. FRAMES

A family of vectors $\{\varphi_k\}_{k \in M}$ in a Hilbert space \mathcal{H} is a *continuous frame* with respect to a measure μ on the index set M if $k \mapsto \varphi_k, M \rightarrow \mathcal{H}$ is measurable and

$$A \|f\|^2 \leq \int_M |\langle f, \varphi_k \rangle|^2 d\mu(k) \leq B \|f\|^2 \quad \forall f \in \mathcal{H}. \quad (1)$$

The constants A and B are called *frame bounds*. If $\{\varphi_k\}_{k \in M}$ is measurable and the upper bound in the above inequality (1) holds, then $\{\varphi_k\}_{k \in M}$ is said to be a *Bessel family* with constant B . A measurable family $\{\varphi_k\}_{k \in M}$ is said to be a *tight frame* if we can choose $A = B$ in (1). If, furthermore, $A = B = 1$, then $\{\varphi_k\}_{k \in M}$ is said to be a *Parseval frame*.

If μ_M is the counting measure and $\Sigma_M = 2^M$ the discrete σ -algebra, we say that $\{\varphi_k\}_{k \in M}$ is a *discrete frame* whenever (1) is satisfied; for this measure space, any family of vectors is obviously measurable.

In the sequel we will use the term *frame* to mean continuous frame. If $\{\varphi_k\}_{k \in M}$ is a frame, then there exists a *dual frame* $\{\psi_k\}_{k \in M}$ which is also a frame with respect to the measure space M such that the reproducing formula

$$f_1 = \int_M \langle f_1, \varphi_k \rangle \psi_k d\mu(k) \quad \forall f_1 \in \mathcal{H} \quad (2)$$

holds weakly, that is,

$$\langle f_1, f_2 \rangle = \int_M \langle f_1, \varphi_k \rangle \langle \psi_k, f_2 \rangle d\mu(k) \quad \forall f_1, f_2 \in \mathcal{H} \quad (3)$$

holds. For discrete frames (2) holds in the strong sense with convergence in norm.

A sequence $\{\varphi_k\}_{k \in \mathbb{N}}$ is said to be a *Riesz sequence* with bounds A and B if

$$A \|\{c_k\}\|_{\ell^2(\mathbb{N})}^2 \leq \left\| \sum_{k=1}^{\infty} c_k \varphi_k \right\|^2 \leq B \|\{c_k\}\|_{\ell^2(\mathbb{N})}^2$$

for all finite sequences $\{c_k\}_{k \in \mathbb{N}}$.

III. CO-COMPACT GABOR SYSTEMS

We let G denote a second countable locally compact abelian group. To G we associate its dual group \widehat{G} which consists of all characters, i.e., all continuous homomorphisms from G into the torus $\mathbb{T} \cong \{z \in \mathbb{C} : |z| = 1\}$. Under pointwise multiplication \widehat{G} is also a locally compact abelian group. Throughout the paper we use addition and multiplication as group operation in G and \widehat{G} , respectively. By the Pontryagin duality theorem, the dual group of \widehat{G} is isomorphic to G as a topological group, i.e., $\widehat{\widehat{G}} \cong G$. Moreover, if G is discrete, then \widehat{G} is compact, and if G is compact, then \widehat{G} is discrete.

We denote the Haar measure on G by μ_G . The Haar measure on any LCA group is unique up to a positive constant. From μ_G we define $L^1(G)$ and the Hilbert space $L^2(G)$ over the

complex field in the usual way. The space $L^2(G)$ is separable because G is assumed to be second countable. For functions $f \in L^1(G)$ we define the Fourier transform

$$\mathcal{F}f(\omega) = \hat{f}(\omega) = \int_G f(x) \overline{\omega(x)} d\mu_G(x), \quad \omega \in \widehat{G}.$$

If $f \in L^1(G)$, $\hat{f} \in L^1(\widehat{G})$, and the measure on G and \widehat{G} are normalized so that the Plancherel theorem holds (see [11, (31.1)]), the function f can be recovered from \hat{f} by the inverse Fourier transform

$$f(x) = \mathcal{F}^{-1}\hat{f}(x) = \int_{\widehat{G}} \hat{f}(\omega) \omega(x) d\mu_{\widehat{G}}(\omega), \quad x \in G.$$

We assume that the measure on a group μ_G and its dual group $\mu_{\widehat{G}}$ are normalized this way, and we refer to them as *dual measures*.

For $a \in G$, the operator T_a , called *translation* by a , is defined by

$$T_a : L^2(G) \rightarrow L^2(G), \quad (T_a f)(x) = f(x - a), \quad x \in G.$$

For $\chi \in \widehat{G}$, the operator E_χ , called *modulation* by χ , is defined by

$$E_\chi : L^2(G) \rightarrow L^2(G), \quad (E_\chi f)(x) = \chi(x) f(x), \quad x \in G.$$

For $t \in L^\infty(G)$ the operator M_t , called *multiplication* by t , is defined by

$$M_t : L^2(G) \rightarrow L^2(G), \quad (M_t f)(x) = t(x) f(x), \quad x \in G.$$

The following commutator relations will be used repeatedly:

$$T_a E_\chi = \overline{\chi(a)} E_\chi T_a, \quad \mathcal{F} T_a = E_{a^{-1}} \mathcal{F}, \quad \text{and} \quad \mathcal{F} E_\chi = T_{\chi} \mathcal{F}. \quad (4)$$

For a subset H of an LCA group G , we define its annihilator as

$$H^\perp = \{\omega \in \widehat{G} \mid \omega(x) = 1 \text{ for all } x \in H\}.$$

The annihilator is a closed subgroup in \widehat{G} , and if H is a closed subgroup itself, then $\widehat{H} \cong \widehat{G}/H^\perp$ and $\widehat{G/H} \cong H^\perp$. These relations show that for a closed subgroup H the quotient G/H is compact if and only if H^\perp is discrete.

For a closed subgroup H of G we let $\pi_H : G \rightarrow G/H$, $\pi_H(x) = x + H$ be the *canonical map* from G onto G/H . If $f \in L^1(G)$, then the function $\dot{x} \mapsto \int_H f(x + h) dh$, $\dot{x} = \pi_H(x)$ defined almost everywhere on G/H , is integrable. Furthermore, when two of the Haar measures on G , H and G/H are given, then the third can be normalized such that Weil's formula

$$\int_G f(x) dx = \int_{G/H} \int_H f(x + h) d\mu_H(h) d\mu_{G/H}(\dot{x}) \quad (5)$$

holds. If, in addition, H is co-compact, i.e., the quotient group is compact, we define the *size* of H as

$$s(H) := \int_{G/H} 1 d\mu_{G/H}(\dot{x}).$$

We will assume the following setup for any given closed, co-compact subgroup H of G . We fix a measure on G and on

G/H . Since G/H is compact, all Haar measures are of the form $c \cdot \tilde{\mu}_{G/H}$, where $c > 0$ and $\tilde{\mu}_{G/H}$ denotes the normalized measure $\tilde{\mu}_{G/H}(G/H) = 1$. With this convention we have $c = d(H)$. By the property of dual measures, this means that the discrete subgroup $H^\perp \subset \widehat{G}$ is equipped with the counting measure multiplied by the constant $1/d(H)$.

Example 2 (lattices). Let $H = P\mathbb{Z}^n$, $P \in \text{GL}_d(\mathbb{R})$, be a subgroup of $G = \mathbb{R}^n$. Clearly, H is both discrete and co-compact, hence a lattice. As measure on $\mathbb{R}^n/P\mathbb{Z}^n$ we pick the normalized measure multiplied with $|\det P|$. With this choice the size of H becomes $d(H) = |\det P|$, and the measure on H is the (normalized) counting measure. Note that $\mathbb{R}^n/P\mathbb{Z}^n$ can be identified with the fundamental domain $F := P([0, 1)^n)$ and that $d(H)$ is its volume. Hence, Weil's formula reads

$$\int_{\mathbb{R}^n} f(x) dx = |\det P| \int_F \sum_{h \in P\mathbb{Z}^n} f(x + h) dx$$

Example 3 (co-compact subgroups). Let $H = \mathbb{R}^s \times P\mathbb{Z}^{d-s} \subset G := \mathbb{R}^n$, where $P \in \text{GL}_{d-s}(\mathbb{R})$ and $s > 0$. The subgroup H is no longer discrete (but it is co-compact) so we cannot choose the counting measure on H . It is therefore natural to equip the compact subgroup \mathbb{R}^n/H the normalized Haar measure. With this choice we see that $d(H) = 1$. The annihilator of H is discrete and is given by

$$H^\perp = \{0\}^s \times (P^T)^{-1} \mathbb{Z}^{d-s}.$$

The Haar measure on H^\perp is simply the (normalized) counting measure.

Setting $F := \{0\}^s \times P([0, 1)^{d-s})$, Weil's formula reads

$$\int_{\mathbb{R}^n} f(x) dx = \int_F \sum_{h \in H} f(x + h) dx$$

From Example 2 and 3 we see that the quantity $d(H)$ generalizes the usual concept of lattice size [12].

Definition III.1. Let $\Gamma \subset \widehat{G}$ and $\Lambda \subset G$ be closed, co-compact subgroups. A *co-compact Gabor system* in $L^2(G)$ with generator $g \in L^2(G)$ is a family of functions of the form

$$\mathcal{G}(g, \Lambda, \Gamma) := \{E_\gamma T_\lambda g\}_{\gamma \in \Gamma, \lambda \in \Lambda}.$$

We need the following characterizations of dual Gabor frames; it follows from a more general result on so-called characterizing equations for generalized translation invariant systems from [13].

Theorem III.2 ([13]). *Let $\Gamma \subset \widehat{G}$ and $\Lambda \subset G$ be closed, co-compact subgroups. Suppose that the two systems $\mathcal{G}(g, \Lambda, \Gamma)$ and $\mathcal{G}(h, \Lambda, \Gamma)$ are Bessel systems. Then the following statements are equivalent:*

- (i) $\mathcal{G}(g, \Lambda, \Gamma)$ and $\mathcal{G}(h, \Lambda, \Gamma)$ are dual frames for $L^2(G)$,
- (ii) for each $\alpha \in \Gamma^\perp$ we have

$$s_\alpha(x) := \int_\Lambda \overline{g(x - \lambda - \alpha)} h(x - \lambda) d\mu_\Lambda(\lambda) = d(\Gamma) \delta_{\alpha, 0}, \quad (6)$$

for a.e. $x \in G$.

For $g, h \in L^2(G)$ assume that two co-compact Gabor systems $\mathcal{G}(g, \Lambda, \Gamma)$ and $\mathcal{G}(h, \Lambda, \Gamma)$ are Bessel systems with bounds B_g and B_h , respectively. By an application of Cauchy-Schwarz' inequality and [13, Proposition 3.3], we see that $s_{g,h,\alpha} \in L^\infty(G)$; to be precise:

$$|s_{g,h,\alpha}(x)| \leq B_g^{1/2} B_h^{1/2} \quad \text{for a.e. } x \in G.$$

Note that the function $s_{g,h,\alpha} : G \rightarrow \mathbb{C}$ is Λ -periodic. Since G/Λ is compact and s_α uniformly bounded, we can consider $s_{g,h,\alpha}$ as a function in $L^2(G/\Lambda)$. Its Fourier series is given by

$$s_{g,h,\alpha}(x) = \frac{1}{d(\Lambda)} \sum_{\beta \in \Lambda^\perp} c_{\alpha,\beta} \beta(x),$$

where the Fourier coefficients $c_{\alpha,\beta}$ are given by:

$$\begin{aligned} c_{\alpha,\beta} &= \int_{G/\Lambda} s_{g,h,\alpha}(\dot{x}) \overline{\beta(\dot{x})} d\dot{x} \\ &= \int_{G/\Lambda} \int_{\Lambda} \overline{g(x - \lambda - \alpha)} h(x - \lambda) \overline{\beta(x - \lambda)} d\lambda d\dot{x} \\ &= \int_G h(x) \overline{\beta(x)} g(x - \alpha) dx = \langle h, E_\beta T_\alpha g \rangle. \end{aligned}$$

In the third equality we have used Weil's formula.

IV. DUALITY RESULTS

For a co-compact Gabor system $\mathcal{G}(g, \Lambda, \Gamma)$ its *adjoint* Gabor system is given by $\mathcal{G}(g, \Gamma^\perp, \Lambda^\perp)$. The time-frequency shifts in a Gabor system and its adjoint system are characterized by the fact that they commute [14, Section 3.5.3], [15, Lemma 7.4.1]. That is, for $(\lambda, \gamma) \in \Lambda \times \Gamma$ the point $(\alpha, \beta) \subset G \times \hat{G}$ belongs to $\Gamma^\perp \times \Lambda^\perp$ if and only if

$$(E_\gamma T_\lambda)(E_\beta T_\alpha) = (E_\beta T_\alpha)(E_\gamma T_\lambda).$$

The first result below shows Bessel duality between a Gabor system $\mathcal{G}(g, \Lambda, \Gamma)$ its adjoint Gabor system. To be more precise, a co-compact Gabor system is a Bessel family if, and only if, the adjoint Gabor system is a Bessel sequence. The result generalizes a result in [14]; a proof can be found in [10].

Proposition IV.1 ([10]). *Let $B > 0$ and $g \in L^2(G)$ be given. Let Γ and Λ be closed, co-compact subgroups. Then $\mathcal{G}(g, \Lambda, \Gamma)$ is a Bessel family with bound B if, and only if, $\mathcal{G}(g, \Gamma^\perp, \Lambda^\perp)$ is a Bessel sequence with bound $d(\Gamma)d(\Lambda)B$.*

A. Wexler-Raz biorthogonality relations

We are now able to state and prove Wexler-Raz biorthogonality relations [3], [6] for co-compact Gabor systems in $L^2(G)$.

Theorem IV.2 ([10]). *Let Γ and Λ be closed, co-compact subgroups. Let $g, h \in L^2(G)$ and assume that two Gabor systems $\mathcal{G}(g, \Lambda, \Gamma)$ and $\mathcal{G}(h, \Lambda, \Gamma)$ are Bessel families. The two Gabor systems are dual frames if, and only if,*

$$\langle E_\beta T_\alpha h, E_{\beta'} T_{\alpha'} g \rangle = d(\Gamma)d(\Lambda)\delta_{\alpha,\alpha'}\delta_{\beta,\beta'} \quad (7)$$

for all $\alpha, \alpha' \in \Gamma^\perp$ and $\beta, \beta' \in \Lambda^\perp$.

Proof. Assume that the two Gabor systems are dual frames. Then, for each $\alpha \in \Gamma^\perp$, we have $s_{g,h,\alpha}(x) = d(\Gamma)\delta_{\alpha,0}$ for a.e. $x \in G$. Recall that the Fourier series of s is:

$$s_{g,h,\alpha}(x) = \frac{1}{d(\Lambda)} \sum_{\beta \in \Lambda^\perp} \langle h, E_\beta T_\alpha g \rangle \beta(x).$$

By uniqueness of the Fourier coefficients, it follows that

$$\langle h, E_\beta T_\alpha g \rangle = d(\Gamma)d(\Lambda)\delta_{\beta,1}\delta_{\alpha,0}$$

for all $\alpha \in \Gamma^\perp$ and $\beta \in \Lambda^\perp$. On the other hand, by unitarity of the translation and modulation operator, we have

$$\langle h, E_\beta T_\alpha g \rangle = \langle E_{\beta'} T_{\alpha'} h, \overline{\beta(\alpha)} E_{\beta'} T_{\alpha'+\alpha} g \rangle$$

using (4). Combining the two last equations yields (7). The converse direction is now immediate. \square

B. The duality principle

The following result on Riesz sequences in abstract Hilbert spaces is a subspace variant of [16, Theorem 3.4.4] and [17, Theorem 7.13]. It is a scaled version of a result in [10] whose proof we follow.

Theorem IV.3. *Let $A, B, c > 0$, and let $\{\varphi_k\}$ be a sequence in a Hilbert space. Then the following statements are equivalent:*

- (a) *$\{\varphi_k\}$ is a Riesz sequence with lower bound cA and upper bound cB ,*
- (b) *$\{\varphi_k\}$ is a Bessel sequence with bound cB and possesses a biorthogonal system $\{\psi_k\}$ normalized so that $\langle \varphi_k, \psi_n \rangle = c\delta_{n,k}$, which is also a Bessel family with bound cA^{-1} .*

Proof. Assume first $c = 1$. Assume that (a) holds. Set $V = \overline{\text{span}}\{\varphi_k\}$. Let $\{\psi_k\}$ be the unique dual Riesz sequence of $\{\varphi_k\}$ in V so that $\overline{\text{span}}\{\psi_k\} = V$. This implies (b).

Assume that (b) holds. Since $\{\varphi_k\}$ and $\{\psi_k\}$ are biorthogonal, it follows that

$$\varphi_j = \sum_{k=1}^{\infty} \langle \varphi_j, \psi_k \rangle \varphi_k$$

for all j . By linearity, we have, for any $f \in \text{span}\{\varphi_k\}$,

$$f = \sum_{k=1}^{\infty} \langle f, \psi_k \rangle \varphi_k.$$

This formula extends to $\overline{\text{span}}\{\varphi_k\}$ by continuity. Now, for any $f \in \overline{\text{span}}\{\varphi_k\}$, we have

$$\begin{aligned} \|f\|^2 &= |\langle f, f \rangle| = \left| \sum_{k=1}^{\infty} \langle f, \psi_k \rangle \langle \varphi_k, f \rangle \right| \\ &\leq \left(\sum_{k=1}^{\infty} |\langle f, \psi_k \rangle|^2 \sum_{k=1}^{\infty} |\langle \varphi_k, f \rangle|^2 \right)^{1/2} \\ &\leq A^{-1/2} \|f\| \left(\sum_{k=1}^{\infty} |\langle f, \varphi_k \rangle|^2 \right)^{1/2}. \end{aligned} \quad (8)$$

We see that $\{\varphi_k\}$ is a frame sequence with lower frame bound A ; by assumption the upper frame bound is B . By the fact that

$\{\varphi_k\}$ possesses a biorthogonal sequence, it follows that $\{\varphi_k\}$ is, in fact, a Riesz sequence with the same bounds.

For $c \neq 1$ repeat the above computations for $\{\frac{1}{c^{1/2}}\varphi_k\}$ and $\{\frac{1}{c^{1/2}}\psi_k\}$ \square

We now state and prove the duality principle for co-compact Gabor systems.

Theorem IV.4 ([10]). *Let $g \in L^2(G)$. Let Γ and Λ be closed, co-compact subgroups. Then $\mathcal{G}(g, \Lambda, \Gamma)$ is a frame for $L^2(G)$ with bounds A and B if, and only if, $\mathcal{G}(g, \Gamma^\perp, \Lambda^\perp)$ is a Riesz sequence with bounds $d(\Gamma)d(\Lambda)A$ and $d(\Gamma)d(\Lambda)B$.*

Proof. Let $\mathcal{G}(g, \Lambda, \Gamma)$ be a frame with bounds A and B . The canonical dual frame $\mathcal{G}(S^{-1}g, \Lambda, \Gamma)$ has bounds B^{-1} and A^{-1} . By Proposition IV.1, the sequences $\mathcal{G}(g, \Gamma^\perp, \Lambda^\perp)$ and $\mathcal{G}(S^{-1}g, \Gamma^\perp, \Lambda^\perp)$ are Bessel families with bound $d(\Gamma)d(\Lambda)B$ and $d(\Gamma)d(\Lambda)A^{-1}$, respectively. By Wexler-Raz biorthogonal relations, these two families are biorthogonal:

$$\langle E_{\beta}T_{\alpha}g, E_{\beta'}T_{\alpha'}S^{-1}g \rangle = d(\Gamma)d(\Lambda)\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}$$

Hence, by Theorem IV.3 with $c = d(\Gamma)d(\Lambda)$, we conclude that $\mathcal{G}(g, \Gamma^\perp, \Lambda^\perp)$ is a Riesz sequence with bounds $d(\Gamma)d(\Lambda)A$ and $d(\Gamma)d(\Lambda)B$.

Conversely, suppose $\mathcal{G}(g, \Gamma^\perp, \Lambda^\perp)$ is a Riesz sequence with bounds $d(\Gamma)d(\Lambda)A$ and $d(\Gamma)d(\Lambda)B$. The dual Riesz sequence of $\mathcal{G}(g, \Gamma^\perp, \Lambda^\perp)$ is of the form $\mathcal{G}(h, \Gamma^\perp, \Lambda^\perp)$ for some $h \in L^2(G)$ and has bounds $(d(\Gamma)d(\Lambda)B)^{-1}$ and $(d(\Gamma)d(\Lambda)A)^{-1}$. Using Proposition IV.1 we see that $\mathcal{G}(g, \Lambda, \Gamma)$ has Bessel bound B . On the other hand, $\mathcal{G}(h, \Lambda, \Gamma)$ has Bessel bound $A^{-1}(d(\Gamma)d(\Lambda))^{-2}$. By Wexler-Raz biorthogonal relations, $\mathcal{G}(g, \Lambda, \Gamma)$ and $\mathcal{G}(d(\Gamma)d(\Lambda)h, \Lambda, \Gamma)$ are dual frames. Since the upper frame bound of $\mathcal{G}(h, \Lambda, \Gamma)$ is $A^{-1}(d(\Gamma)d(\Lambda))^{-2}$, it is immediate the upper frame bound of $\mathcal{G}((d(\Gamma)d(\Lambda))^{-1}h, \Lambda, \Gamma)$ is A^{-1} . Hence, by a computation as in (8), we see that A is a lower frame bound for $\mathcal{G}(g, \Lambda, \Gamma)$. \square

Since a Riesz sequence with equal upper and lower bounds is an orthogonal sequence, we have the following corollary of Theorem IV.4.

Corollary IV.5. *Let Γ and Λ be closed, co-compact subgroups. A Gabor system $\mathcal{G}(g, \Lambda, \Gamma)$ is a tight frame if, and only if, $\mathcal{G}(g, \Gamma^\perp, \Lambda^\perp)$ is an orthogonal system. In these cases, the frame bound is given by $A = \|g\|^2(d(\Gamma)d(\Lambda))^{-1}$.*

Example 4. In $L^2(\mathbb{R}^d)$ we consider the co-compact Gabor system

$$\mathcal{G}(g, \Lambda, \Gamma) \quad \text{with } \Lambda = A(\mathbb{R}^s \times \mathbb{Z}^{d-s}), \quad \Gamma = B(\mathbb{R}^r \times \mathbb{Z}^{d-r})$$

for some choice of $0 \leq r, s \leq d$. The duality principle says that $\mathcal{G}(g, \Lambda, \Gamma)$ is a continuous frame with bounds A and B , if and only if the adjoint system

$$\mathcal{G}(g, (B^T)^{-1}(\{0\}^r \times \mathbb{Z}^{d-r}), (A^T)^{-1}(\{0\}^s \times \mathbb{Z}^{d-s}))$$

is a Riesz sequence with bounds A and B . If $s = r = d$, the adjoint system is simply $\{g\}$ which is a Riesz sequence with bound $\|g\|^2$. Hence, in this case, the Gabor system $\mathcal{G}(g, \Lambda, \Gamma)$

associated with the short-time Fourier transform is a tight frame for any $g \neq 0$.

Remark 1. By choosing the normalized measure on G/Λ and \widehat{G}/Γ , we have that $d(\Lambda) = d(\Gamma) = 1$. Hence, in this case, the Wexler-Raz relations become a biorthonormality condition, while in the duality principle we have preservation of bounds between the frame and the Riesz sequence. This shows that choosing normalized measures on G/Λ and \widehat{G}/Γ in some sense is very natural. On the other hand, this choice does not agree with the standard choice of measures in lattice Gabor analysis.

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